BACKPAPER EXAMINATION ALGEBRAIC GEOMETRY

Answer ANY SIX questions, all questions carry equal marks. Assume that the base field k, in all questions below, is algebraically closed. Time: 3 hours. Total Marks: 100

(1) Let $C = V(F) \subset \mathbb{A}^2_k$ be an irreducible plane affine curve (where $F \in k[x, y]$) and let $P \in C$ be a point of C. Define the local ring $\mathcal{O}_P(C)$ of C at P. Define the notion of a discrete valuation ring. Show that if P is a simple (or smooth) point of C, then $\mathcal{O}_P(C)$ is a discrete valuation ring.

(2) State Bezout's Theorem and define all the terms that appear in the statement. Verify the statement of Bezout's Theorem for the two curves $F(x, y, z) = y^2z - x(x - z)(x + z)$ and $G(x, y, z) = y^2z - x^3$.

(3) Define the notion of a line in $\mathbb{P}^2_k$. Show that there is a bijection between the set of all points of $\mathbb{P}^2_k$ and the set of all lines in $\mathbb{P}^2_k$.

(4) Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be two collections, each consisting of three points of $\mathbb{P}^2_k$, such that neither collection of three points lie on a line. Prove that there exists a projective change of coordinates T such that $T(P_i) = Q_i$ for $i = 1, 2, 3$. Let L_1, L_2, L_3 and M_1, M_2, M_3 be two collections, each consisting of three lines of $\mathbb{P}^2_k$, such that neither collection of three lines pass through a common point. Show that there exists a projective change of coordinates S such that $S(L_i) = M_i$ for $i = 1, 2, 3$.

(5) Prove that upto projective equivalence, there is only one irreducible conic in $\mathbb{P}^2_k$ (with equation $y^2 = xz$). Show that this curve is nonsingular.

(6) Let $C \subset \mathbb{A}^2_k$ be an affine plane curve, let P be any point on C (not necessarily, a simple point). When is a line L called a tangent line to C at P? Define the notion of the multiplicity $m_P(C)$ of C at P? Show that a line L is tangent to C at P, if and only if, $I_P(C, L) > m_P(C)$.

(7) Define the notions of an irreducible and a nonsingular plane projective curve. Show that any nonsingular plane projective curve is irreducible. Is the converse true?

(8) Let $f : X \to Y$ be a morphism of affine varieties. Define the induced k-algebra homomorphism of coordinate rings $f^* : \Gamma(Y) \to \Gamma(X)$. Show that $f(X)$ is dense in Y, if and only if, $f^* : \Gamma(Y) \to \Gamma(X)$ is injectve.

(9) Let W be the set of all conics in $\mathbb{P}^2_k$. Show that W can be identified with $\mathbb{P}^5_k$. Let P_1, P_2, P_3, P_4 be four distinct points of $\mathbb{P}^2$. Let V be the subset of all conics in $\mathbb{P}^2_k$ passing through these four points. Show that V is a linear subspace (i.e., defined by homogenous linear equations) of $W (= \mathbb{P}^5_k)$. Show that $\dim(V) = 2$ if these four points lie on a line, and $\dim(V) = 1$ otherwise.

(10) Let $C \subset \mathbb{P}^2_k$ be a nonsingular projective plane curve of degree 3, let $O \in C$ be a fixed (but arbitrary) point. Describe the group law on the points of C (such that O is the identity). What is the inverse of a point $P \in C$ with respect to this group law?

(11) Define the notion of birational equivalence between two varieties. Show that $\mathbb{P}^1_k \times \mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are birationally equivalent but not isomorphic.